Fractions, Fraction Ops & Decimals Overview Lesson

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1. Introduction

Workshop's Goals

The lesson subject today is fractions and fraction operation. The goal is for you to understand fraction better, to have alternative perspectives and even give you specific classroom activities so that:

- You will teach fraction more effectively — your students will deeper/better understanding of fractions and more of them will have this knowledge.
- You will teach fraction more efficiently — your students will accomplish it in less time. (Time is the most precious resource in the classroom.

Teachers' Grade level

As far as I am concerned teachers are divided into three categories depending on the knowledge level of their students of a specific subject, in this case fractions and fraction operations:

(a) Preparing to learn the topic;

(b) Learning the topic

(c) Learning new material based on the topic, improving topic comprehension, reviewing and practicing skills.

With respect to fractions, K–3 grades teachers prepare, 3–6 grade teachers teach and 6–8 grade teachers follow-up.

This workshop is designed for all of these teachers. Clearly it addresses directly issues that teachers face teaching fractions and subjects that are based on understanding fractions in grades 3–8. But it also show K–3 teacher how to better prepare their students to understand fractions.

2. Creating Fractions

Introductory Demo & Activities

- **Tear a Sheet of Paper Provocatively.**
- **Instruct:** Tear your own sheet of paper.
- **Question:** Can anyone tell me what did you just do?
  - **Answer:** Created parts of a whole.
  - **Question** (follow-up question): Is there a different between the two parts of the sheet of paper?
    - **Answer:** some of you made equal or almost equal parts, others did not.

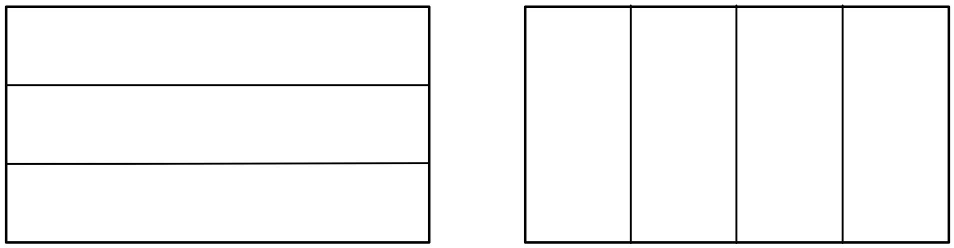
Definition. *When all the parts of the whole are equal to each other we call the parts* ***fractions****.*

Creating Fractions

- Folding paper
  - Estimating folds by other numbers — fold into ***3, 5*** and ***7*** parts.
  - Folding in half repeatedly of the original sheet that was folded by ***2, 3, 5*** and ***7***.
- Cutting paper safely
  - Draw a line along a straight edge many times till the paper tears.
  - For older students (requires dexterity)
    - Fold a sheet sharply and cut.
    - Cut along the edge of a ruler.
- Snap cubes or Letzi
- Geo boards

Using the quadrille graph paper, snap cubes or geo board, create two identically rectangles. Then divide each into equal parts:

- Horizontally
- Vertically.



(a) (b)

Figure 1. Division of Two Identical Single-Unit Rectangles
(a) Divided into 3 equal parts and
(b) Divided into 4 equal parts

3. Important Message: <u>Math. Organization & Precision</u>

☞ [Put on your teacher hat] Math cannot depend on the ability of whoever practices it. Any specific mathematical activity should have the same result whether it is performed by a fourth grader or a PhD mathematician. ***Mathematics is independent of your skill, dexterity, the tools you use and how you use them.***

- But… Knowing how to use the tools correctly and understanding the language — a 2D language, very orderly that abhors ambiguities (nevertheless it has some) — minimize mistakes. Therefore, ***practicing math, we strive to be as precise as possible:***
  - Insist on using quadrille graph paper, pencils & erasers
  - To draw points use intersecting lines (tick marks) not dots.
  - Insist on order and organization

    Being organized and precise does not improve the math, it makes it easier to learn and understand math.

- When teaching a task, isolate the skills necessary to perform the task, but which do not contribute to the learned lesson. ➔ Give alternative tools to any student who straggles with these skills.

  <u>Example</u>. We will work with:

  - ♦ **Quadrille graph paper** — requiring drawing and cutting straight lines.
  - ♦ **Snap cubes** — provide straight lines. Its drawback is that student may be distracted by playing with the cubes.

<u>Math Organization</u>

- The mathematical language is two-dimensional. Ordinary fraction notations are 2–D requiring special procedures and organization.
- The place-value notation is 1–D notation of fractions, making it simpler to work with as all the procedures that apply to integers can be extended to fractions.

4. <u>Fractions Are Relative to Whole Units</u>

Whole units may differ in their…

- **Sizes.** Different pizzas have sizes but all can be partitioned.
- **Shapes.** Apples, pizzas, sheets of paper, groups of kids all have different shapes yet all can be split into different parts.

Demonstrations

Use snap cubes to demonstrate differences in…

- **Size.** Cut in half
  — A full size sheet of paper ➔ a set of 2 halves.
  — Smaller sheet ➔ a set of 2 halves.
  State: Note that, now that accept two equal parts that make the whole as ***halves***, we intuitively recognize both sets of parts as pairs of halves.

- **Shapes.** Use the previously-made snap cubes constructs of 8 cubes to show that one 1/8 is the same regardless of which shape it came from.
  - ***2x2x2***
  - ***1x8***
  - ***2x4***
  - ***4x2***
  - 2D zigzag
  - 3D zigzag

5. The Notation

☞ [Put on your teacher hat] With the fraction notation is the first time students are introduced to the ***two-dimensional property*** of the mathematical language. By this time students are usually experienced with writing numbers above each other, aligning the place values. This 2-D format makes computation easier and reduce the chance for errors. But the numbers and the operations are still read sequentially just as when they were written on a single line. On the other hand, it is **necessary** to write ordinary fractions (not decimals) in a 2-D format.

If you doubt the fact that this adds complexity, just consider what happens when students learn mixed numbers. How often students confuse the integer part with the fraction part and misread the numbers?

So it is necessary to introduce the fraction notation in an intuitive manner, making it easier to remember, understand and use.

A written fraction consists of three components:

The Whole or Wholes	-------➤	**1**
The Splitting Action	-------➤	———
The Outcome Count of Parts	-------➤	**4**

Figure 2. The Fraction Notations — Splitting a Whole Makes a Fraction

☞ [Put on your teacher hat] The whole that is partitioned can be an aggregate of several individual whole units. (See detailed discussion Multiple of Unit Fractions and Unit Fraction of Multiple Wholes.)

6. The Unit Fraction — Introduce the Grid System

Participants form teams of three, working together and sharing tasks and alternating the tools:

- One member works with graph paper, drawing & cutting.
- One member works with snap cubes
- One member works with a geo boards. Using different colors of the rubber bands makes it easier to distinguish different fractions.

Create $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, etc.

☞ **Advantages of the Grid Method** [Put on your teacher hat]

- Every fraction can be created relatively easy and accurately. The only limitation is the size of the grid.
- The grid makes it easy to identify at least three different unit fractions for every rectangle.
- Even when imprecision is present, the grid discourages students from misarranging fractions (as can be easily done with circular fractions Show fraction game that uses circular fractions. Kids make these mistakes even with Montessori fraction inserts.).

7. Multiple of a Unit Fractions = Unit Fraction of Multiple Wholes

Participants form teams of three, working together and sharing tasks and alternating the tools:

Using quadrille graph paper and snap cubes, create $\frac{3}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, etc. by

- Creating multiple of the unit fractions.
- Creating the unit fractions of multiple wholes.

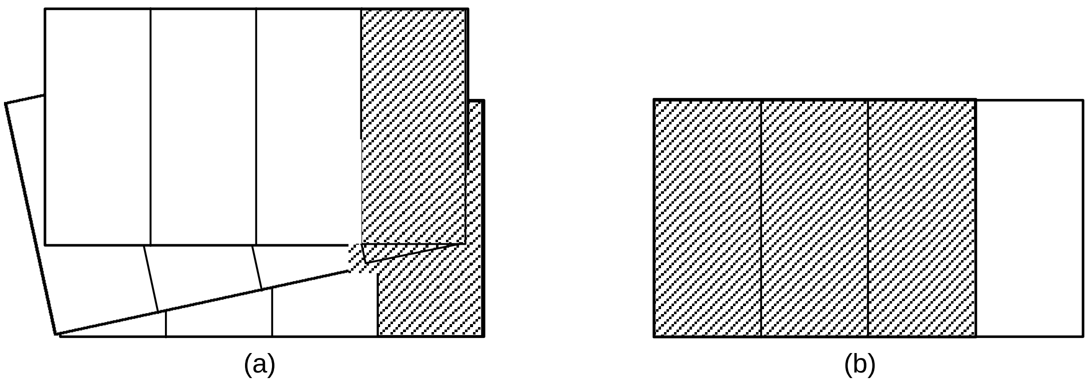


Figure 3. A Quarter of a Triple-Unit Equals Three Quarters of a Single Unit.
(a) Triple-Unit is folded in three so it can be cut.
(b) Three-Quarters Cut from a Single Unit.

Observation. *A part of multiples equals multiple of parts.* Or, expressed mathematically:

A fraction of multiples equals multiple of fractions.

Reason. *A fraction is the result of division and division is the inverse of multiplication.*

Question: Can you tell the difference between

$$3 \text{ of } \frac{1}{4} \text{ of one whole}$$

and

$$\frac{1}{4} \text{ of } 3 \text{ wholes ?}$$

Answer: There is no difference. When the parts are presented, there is no way to tell them apart or what is the source of either.

☞ **Exploit this fact.** [Put on your teacher hat] We can view the denominator as either:

- As a multiplier of a unit fractions; or,
- As a unit fraction of multiple wholes,

Depending not only on the mathematical context but also on which is more comprehensible to individual students.

Creating Any Fraction on the Number Line

☞ [Put on your teacher hat] This is not part of the California Standards but learning it will improve mathematical comprehension of both the teachers and the students.

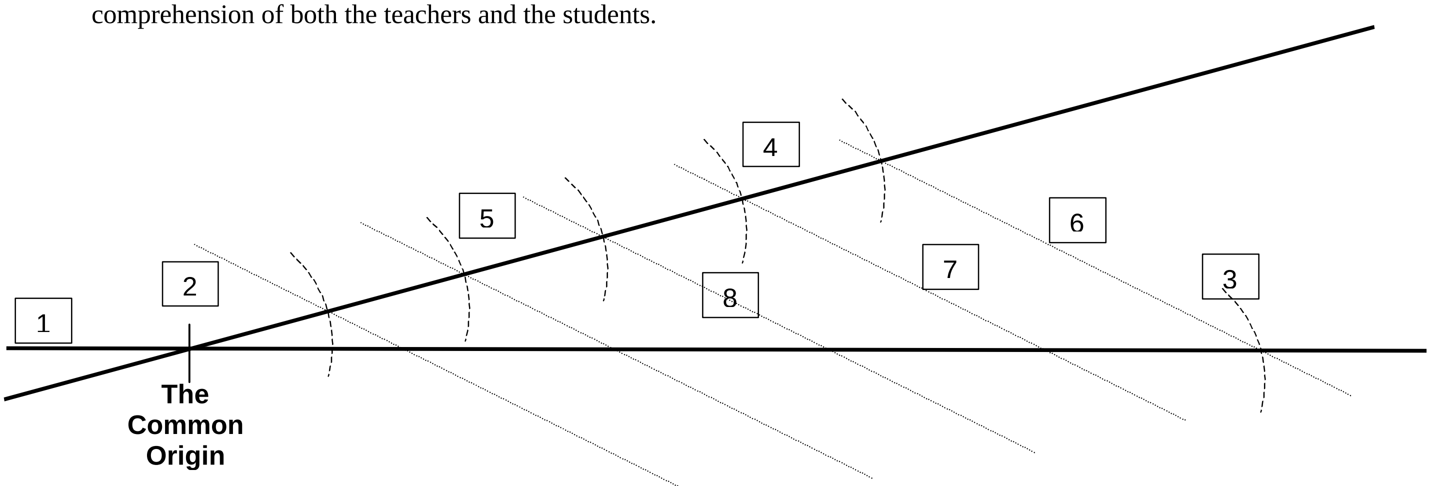


Figure 4. Dividing a Line Segment to a Desired Count of Equal Sub-segments

Informal Procedure

If you want to create a specific fraction, such as ***1/9***, ***3/7*** or ***7/5*** on the number line, do the following:

1. Draw the line you want to partition. This is your *target line.*
2. Draw the origin on this line.

3. From the origin draw mark on the line the length that you wish to divide.

4. Draw a second line such that it intersects with the first line in its origin. This is your *measuring line.*

5. Divide the measuring line into equal number of segments. The number of segments is equal to the number of divisions you need in order to create the desired fractions. For example, if you wish to create a ***3/7***, you need to draw ***7*** segments on the second line.

6. Draw a line from the end of the last segment on the measuring line to the end mark on your target line. This is your *gauge line.*

7. Now draw lines such that
   - Each line starts at the end of a segment on the measuring line;
   - Each line intersects the target line; and
   - Each line is parallel to the gauge line.

The target line is now divided into the desired unit fractions. If you wanted to create a fraction that is a multiple of the unit fraction, select the desired number of contiguous segments. For example, if you wish to create a ***3/7***, the target line is now divided into ***7*** segments, each of which is ***1/7*** long of the target line. Any contiguous ***3*** segments represent ***3/7*** of the target line.

Formal Procedure for Extra Credit

The number line can be used to generate a fraction, say fifths? — The following procedure enables you to divide a Line Segment to N Equal Sub-segments.

1. Draw the target line that you with to divide.
2. Mark on it the origin and call it ***O.***
3. Starting at its origin, mark on it the desired segment length. Mark this end point by ***A.***
4. Draw an arbitrary number line in an arbitrary angle to the target line, such that the two lines intersect at their origin. That is, the origins of both lines coincide.
5. Mark on this number line as many units equal to the number by which you wish to divide the target line. If this number is ***n***, mark the end point of this number line with ***N.***
6. Draw a connecting line from ***N*** on the number line to ***A*** on the target line. This line ***NA*** is the base of the triangle ***OAN.***
7. Now draw a line parallel to ***NA*** from the point ***N-1*** till it intersects with the target line.
8. Repeat step 7 for all the numbers ***1, 2, 3, … N-2.***

The target line segment is now divided to exactly ***N*** equal sections.

8. Modeling Fractions

We can model fractions using identical objects, such as paper clips or snap cubes, from which we can create whole that consist of equal parts.

The Number Chain

1. Decide on a number ***n***, where ***n*** is the number of paper clips that represent ***1***, the single unit.

2. Use ¼" masking tape (for many practical reason it works best but you can use any other tape) to tape together several sets of ***n*** paper clips.

   2.1. You can use markers to color the tape (or use the more expensive colored tape to begin with). [See Number Line from Snap Cubes below.]

3. Connect these paper-clip sets together.

4. Label the fraction clips after the end of each. Follow the rules of labeling integers. You can use the connecting tape for labeling the fractions. Additional labels are needed where there links are lose.

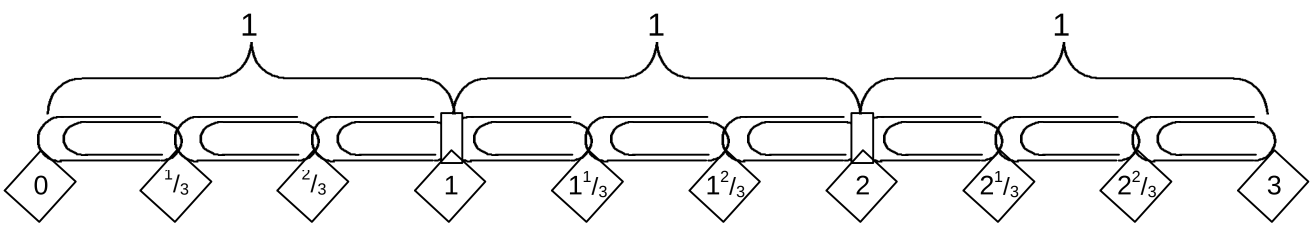


Figure 5. Fraction Line from Number Chain

☞ [Put on your teacher hat] Note. In this discussion we do not have examples. Instead we employ the mathematical convention to represent numbers with letters. If this approach is used with students, it prepares them for general mathematical concepts such as those used in algebra.

Number Line from Snap Cubes

1. Decide on the number ***n***, where ***n*** is the number of cubes that represent ***1***, the single unit. Color selection:

   1.1. You can select that each cube in the set of ***n*** cubes is of a different color; or,

   1.2. You select all cubes of the same set to be of the same color, different sets having different colors.

2. Label the fraction cubes. Follow the rules of labeling integers.

Fractions of Fractions

1. Decide on the numbers ***n*** and ***m*** cubes, where ***n*** is the number of integers you want to have and ***m*** is the number of parts that represents ***1***, the single unit.

2. Color selection. For the first ***m*** cubes select a color pattern such that each cube has a different color. That is each cube represents the fraction $^{1}/_{m}$. (After the experienced students can experiment with other color schemes with multiple cubes having the same color.)

3. Repeat step 2 ***n*** times, for each set of ***m*** cubes, using the same color scheme.

4. Snap together the ***n*** sets of ***m*** cubes. The consistent and repeating color pattern should be apparent.

This model can be used to represent ***a fraction of a fraction*** or ***fraction multiplication***. For we can consider the complete sequence of ***n*** sets of cubes as the single unit, or ***1***. In this view, a single set is then $^{1}/_{n}$ and a single cube is then $^{1}/_{m}$ of $^{1}/_{n}$, or $^{1}/_{m} \times {}^{1}/_{n}$.

9. Basic Fraction Operations with the Grid Method

The grid method exploits any set of tools, with which a grid can easily be generated, to create whole units and then divide them into at least three sets of fractions. Such tools are:

- Quadrille graph paper (square grid), straight edge, pencil (colored are recommended), eraser and, optionally, scissors or other paper-cutting means;
- Geo boards and rubber bands;
- Snap cubes;
- Lego or Duplo (square bricks).

Participants work in teams of three, using graph paper, snap cubes and geo board. Team members alternate roles as they perform operations on different pairs of fractions.

- ♦ One member works with graph paper, drawing & cutting.
- ♦ One member works with snap cubes
- ♦ One member works with geo boards:
  - Use a single geo board, if it is has a large grid; or,
  - Use 3 boards, when using the common ***4x4*** boards.

  Using different colors of the rubber bands makes it easier to distinguish different fractions.

☞ [Put on your teacher hat] **Warning: Do not tell the students how to figure out what is the common fraction.**

- Guide them to discover it on their own.
- Guide them to associate the ***name of a fraction*** with the ***number of parts*** into which the whole is partitioned.
- Guide them to generalize the **fraction naming rule** with respect to a **single grid square** that is shared by both partitions (the common denominator).

After the students performed these activities a few times, they are proficient with the procedures and familiar with the informal terminology. Now we can ask questions that lead them to the formal terminology..

Question. What did we just do?

Answer. We compared, subtracted and added fractions.

☞ [Put on your teacher hat] The students performed the following operations in this order:

1. **Fraction comparison.** Compare the magnitude of two fractions;
2. **Fraction subtraction.** Determine the magnitude of the difference between the two fractions. That is, subtract the smaller fraction from the larger one;
3. **Fraction addition.**

Note that the students accomplish these operations although

- Fraction terminology is yet to be introduced;
- The students discover the concept of common denominator on their own.

Comparing, Subtracting and Adding Fractions with the Grid Method

For detailed instructions see the document Fraction Operation with the Grid Method.

10.Fraction Multiplication

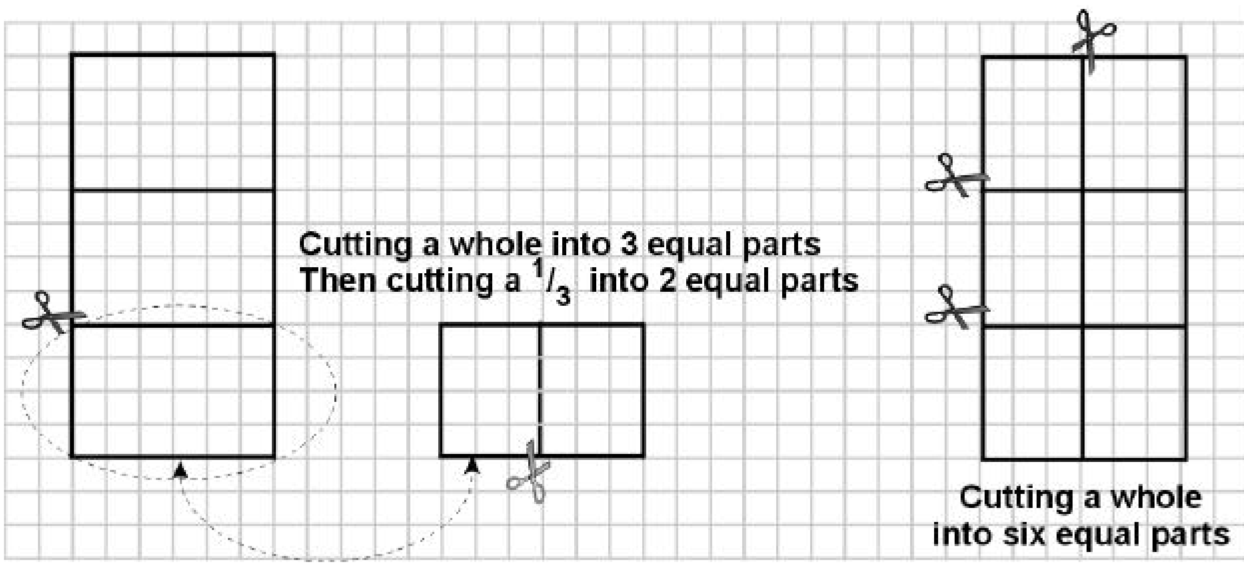


Figure 6. Multiplying $^1/_3$ by $^1/_2$ visually on graph paper

Question. What is the difference between halving a single $^1/_3$ and halving a whole that was divided into thirds?

Answer. A half of a third is sixth. Halving a whole that was partitioned into thirds yield six sixths. Either was, if the original wholes were equal, then the sixths are equal.

☞ [Put on your teacher hat] **Note.** Using words rather than numbers reinforces the notion that we can refer to fractions as objects that can be counted, like apples, and then can be multiplied.

- A brief intro to multiplication — repeated addition vs. unit multiplication
- Visualize on a graph
- Practical activities

Figure 7 and Figure 8 illustrate two different graphic interpretations of fraction multiplication.

- **Figure 7.** The result is a new object or ***unit***. When we multiplying a *linear unit* by a *linear unit* the product is expressed with a new unit, the ***area unit.***
- **Figure 8.** The product of the multiplication is expressed in terms of the same units of the multipliers. For the equation for calculating this product, denoted by ***X,*** is:

$$\frac{X}{\frac{1}{2}\,\text{units}} = \frac{\frac{3}{4}\,\text{units}}{1\,\text{units}} 5$$

Therefore,

$$X = \frac{\frac{3}{4}\,units}{1\,units} \times \frac{1}{2}\,units = \frac{3}{4} \times \frac{1}{2}\,units$$

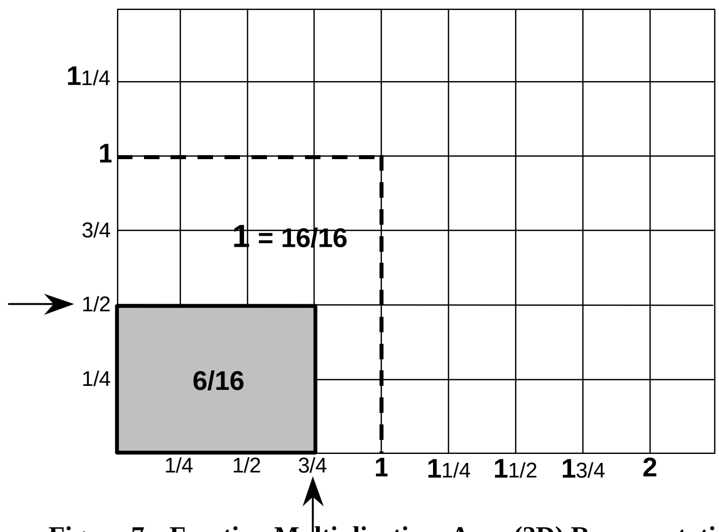


Figure 7 Fraction Multiplication, Area (2D) Representation.
3/4 x 1 /2 = 6/16 = 3/8 of the Area

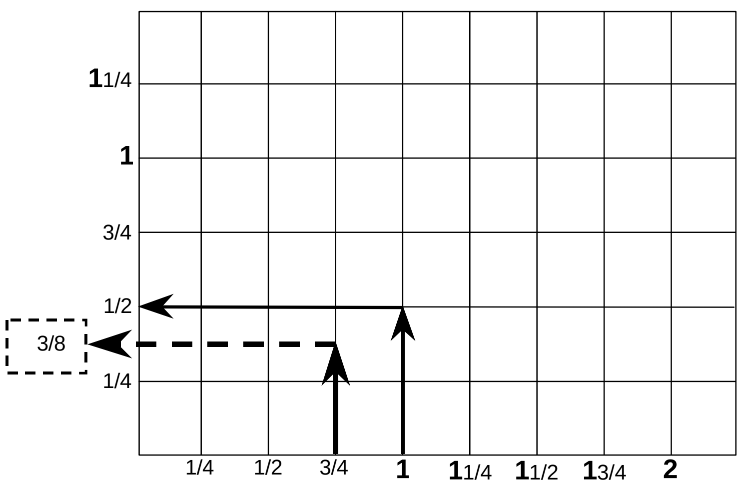


Figure 8. Fraction Multiplication, Linear Representation.
$^{3}/_{4}$ x ½ = $^{3}/_{8}$ of the Perpendicular Line

Alternative View: Considering Fraction Multiplication as Unit Multiplication

Unit multiplication can be use in the context of numeric multiplication, when multiplication is nothing but repeated addition. This approach gives teachers and students a different perspective on multiplication. For example, consider

4,000 x 5,000

as

4 thousands x 5 thousands

just as we do ***4 feet x 5 feet***. Similarly, just as ***foot x foot = foot***^{2} we need to now what is ***thousand x thousand***? Once the students know that ***thousand x thousand = million***, then

4 thousands x 5 thousands
= 4 x 5 x thousands x thousands
= 4 x 5 x million
= 20 x millions

This same techniques can be applied to fractions. For example, consider

$$\frac{2}{3}\times\frac{5}{7}$$

Look at $^2/_3$ and $^5/_7$ differently. We can represent the multiplication as

$$=\left(2\times\frac{1}{3}\right)\times\left(5\times\frac{1}{7}\right)=$$

where the unit fractions are the "units" (just like apples or thousands) and the numbers **2** and **5** represent how many things we have of each, respectively. This approach leads us to:

$$=2\times5\times\left(\frac{1}{3}\times\frac{1}{7}\right)=$$

The only new thing we have to figure out now is the answer to the question: What is the unit that is the product of multiplying the units $^1/_3$ by $^1/_7$? The answer to this question, $^1/_{21}$, is derived from the previous discussion of fraction multiplication.

Therefore, we get

$$=2\times5\times\frac{1}{21}=10\times\frac{1}{21}\text{ which we write as }\frac{10}{21}.$$

The Inverse Relationship Between Integers and Unit Fractions

We know that positive and negative numbers are inverse of each other. In this respect, ***zero*** is the constant or reflection point on the number line. ***Zero*** is also the identity number with respect to addition and subtraction. That is, the outcome of adding a zero to any number, ***n***, or subtracting a zero from any number, ***n***, is the same number ***n***. This leads us to conclude that we can do away with subtraction and have; ***Having both positive and negative numbers addition suffices.***

Similarly, fractions are the inverse numbers of integers and vice versa with respect to multiplication and division and the number one is their identity number and point of reflection on the number line. However, it is important to note that the symmetry between integers and fractions is not linear as it is between positive and negative numbers.

11. Fraction Division

Integer Division Revisted

There are several ways to answer the questions

What is division?

And paraphrasing:

What do we do when we divide one number by another?

The common answers are usually with respect to dividing a set of objects between several recipients. This sort of representation views division as the operation that answers such questions as

*If we have **n** items and we wish to divide them into **m** groups, how many items will end in each group?*

An example is when we need to divide **28** desk between **4** classrooms.

However, this representation is not helpful when considering dividing by fractions. In particular it is counter-intuitive when trying to explain dividing one fraction by another. For example, consider dividing a pizza among several kids. We easily understand dividing 3 pizzas among 5 kids and even dividing $^4/_5$ of a pizza among 3 kids. But can we divide ½ a pizza among a $^1/_3$ of a kid?

An alternative view of division is to think of it as answer the question

*How many groups of **m** items each, will fit into the total of **n** items?*

For example, if we need to divide **28** desk among several classrooms and each classroom should receive ***7*** desks, how many classrooms can we furnish?

In general, the question can be state informally as:

How many of <u>this</u> fit or can get into <u>that</u>?

where both *this* and *that* are integers.

Introducing Fraction Division

Now consider that both *this* and *that* stand for fractions. Mathematically speaking, we can state the question:

*How many of **m/n** fit or can get into **p/q**?*

where ***m, n, p*** and ***q*** are integers and $n \neq 0, q \neq 0$.

For example, if a coach instructs a runner run $^4/_5$ of a mile but the school track is only $^1/_4$ of a mile long, how many times she must run around the track to complete her assignment? In other words, the question is how many $^1/_4$-mile units fit in a $^4/_5$ mile?

We proceed teaching how to divide a fraction by a fraction by examining several examples.

- Practical activities — torn page repair with scotch tape.

☞ [Put on your teacher hat] Practical examples of fraction division often involve dividing one measurement by another.

Dividing Measurements. Unit division plays an critical part with fraction division.

- ♦ **When the two measurements use different units we are dealing with measuring rate of change. The most common of these is speed — miles per hours.**
- ♦ When the two measurements use the same units the result is **a plain number with no units**.

Example 1: Fixing Torn Sheet of Paper

☞ [Put on your teacher hat] Achieving the objective of the lesson successfully depends on the quality of the materials used by the students. For example, measurements should be correct. Therefore, the students should prepare the materials for this lesson on their own only if there are capable to do so.

Matrials

☞ [Put on your teacher hat] If the precise materials described below are not available, you may adapt the lesson according to what is available to your students.

- A sheet of graph paper (for our example we use 4 squares per inch) that is sliced as follows:
    - Perpendicular to one edge make a precise **¾**-inch long slice.
    - Perpendicular to one edge make a precise **½**-inch long slice.
- Cut a $^1/_4$-inch wide masking take into several strips that are **½**-inch and **¾**-inch long.

Activity 1

Instruct the students to use the **½**-inch long strips of tape to mend the **¾**-inch long paper cut. They should follow these rules:

- Use each strip of tape lengthwise.
- If more than one strips is needed, they should place the strips next to each other with care to avoid gaps and overlaps.
- If a piece is too long, that is, if a strip extends beyond the end of the cut, they should trim it.

Question to the students:

How many strips of tape, each of which is ½-inch long, are needed to mend the ¾-inch long tear?

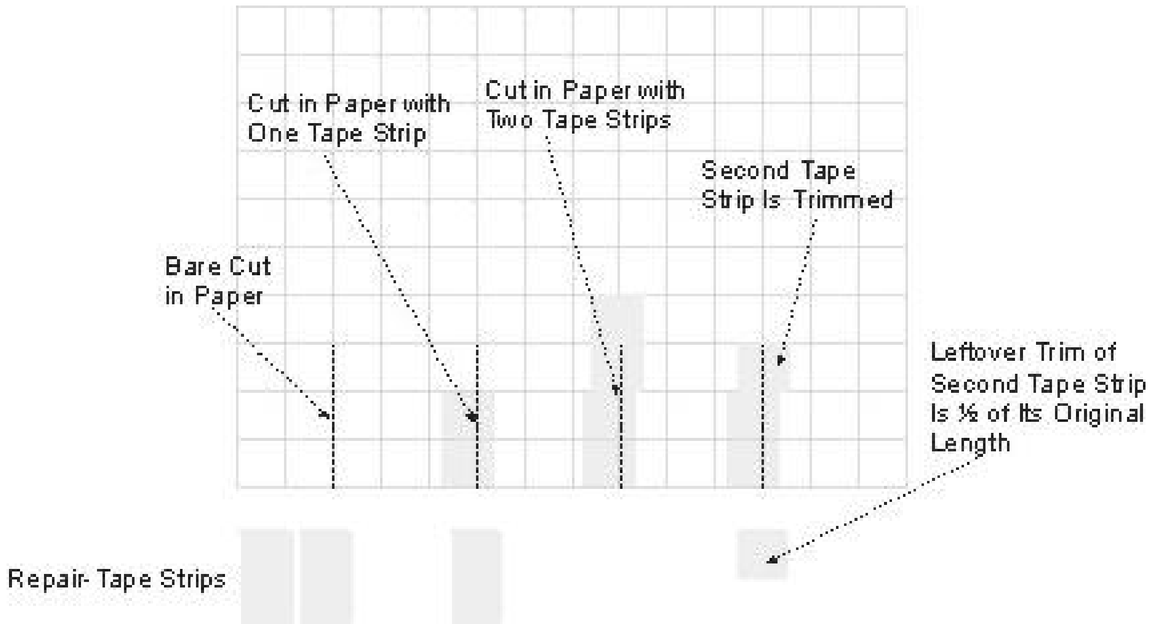


Figure 9. Divide ¾ by ½.

Activity 2, Inverse of the Task Presented in Activity 1

Instruct the students to use the **¾**-inch long strips of tape to mend the **½**-inch long paper cut, following the same rules as in the previous exercise.

Question to the students:

How many strips of ¾-inch long tape are needed to mend the ½-inch long cut?

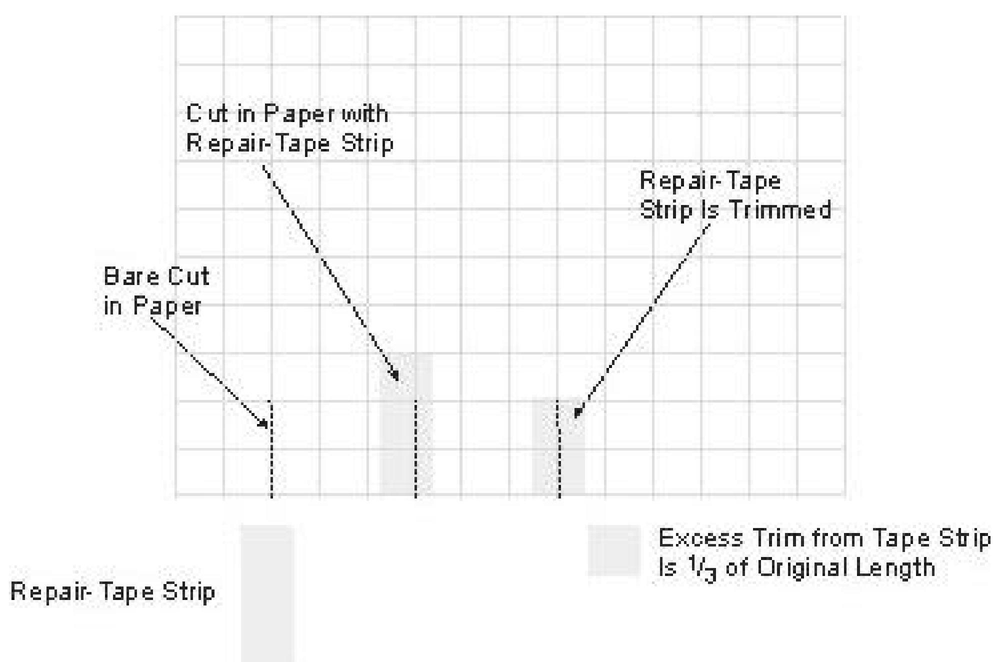


Figure 10 Divide ½ by ¾.

☞ [Put on your teacher hat] The following mathematical discussion is too advanced and may not be suitable for an introductory lesson. It is given here for the benefit of the teacher. It must be modified to suit the knowledge of the students.

☞ [Put on your teacher hat] Note that discussion bellow employs the reciprocity relationship between integers and unit fractions. Students should be familiar with it by this phase of their fraction learning. This is approach, which is founded on integer division, is mathematically sound. It is preferred over the alternative approach, which employs dividing both numerator and denominator by the same value.

To see how this problem is handled numerically, let's translate its visual representation into numerical one and handle it the same as we did for the previous exercise. Generally, it asks

How many three-quarters equal one half?

Numerically it is

$$\frac{1/2}{3/4} = \frac{1/2}{3/4} x 1 = \frac{1/2}{3/4} x \frac{4/3}{4/3} =$$

Again we use our knowledge of the inverse rule and select $^4/_3$ to be the number from which we construct the ***1***.

$$= \frac{1/2}{\cancel{3/4}} x \frac{4/3}{\cancel{4/3}}$$

As before, based on fraction multiplication, the ¾ and the $^4/_3$ (the two denominators) cancel each other and we are left with

$$= \frac{1}{2} x \frac{4}{3} = \frac{2}{3}$$

Again the numeric result coincides with our experience. To our satisfaction this result matches precisely our practical observation.

Note that this problem is the inverse of the one posed in prior exercise. Multiplying their results yields ***1***.

Example 2, Simulation — Food Purchase

Source Note. This is a variation on an example, which Ruth Parker [mec@mec-math.org] often uses.

Ruth bought ***5*** identical slices of turkey meat totaling $^1/_3$ lb. She prepared a sandwich on which she wanted to place ¼ lb. of the meat. How much of the slices she used?

Computational Solution

The mathematician's solution is a formalization of the following intuitive consideration:

If $^1/_3$ lb. consists of ***5*** slices, then ***1*** lb. consists of ***15*** slices.

Since she is limited to ¼ lb., then ¼ of ***15*** slices is ***3 ¾*** slices.

Formally: Let $\boldsymbol{x}$ = Number of slices in 1 lb.
$\boldsymbol{y}$ = Number of slices she may eat.

Figuring the number of slices per pound (the $\boldsymbol{x}$):

1) $$\frac{x}{5} = \frac{1}{1/3}$$

We can multiply both sides of the equation by the same number without disturbing it. The result is

$$\frac{x}{5} \times 5 = \frac{1}{1/3} \times 5$$

The two **5**'s on the left side of the equation cancel each other. The result is

$$x = \frac{5}{1/3} = \frac{5}{1/3}$$

Following the procedure we developed in the previous example:

$$= \frac{5}{1/3} \times 1 = \frac{5}{1/3} \times \frac{3}{3}$$

As before the 2 denominators cancel each other resulting with

$$= 5 \times 3 = 15$$

Figuring the number of slices she may eat (the $\boldsymbol{y}$):

2) $$\frac{y}{1/4} = \frac{x}{1}$$

Again multiply both sides by the same number without disturbing it:

$$\frac{y}{1/4} \times 1/4 \quad \frac{x}{1} \times \frac{1}{4}$$

$$= \frac{1}{4} \times x$$

$\boldsymbol{x}$ from (1) in (2), we get

$$y \quad \frac{1}{} \times 15 = \frac{15}{} \quad 3\frac{3}{}$$

Visual Solutions

This time we use the visual simulation — students can actually perform it, using model materials — to check if our numeric calculation is correct. Since ***3*** slices make a $^1/_3$ of a pound, we can quickly arrive (deduce) that ***9*** slices make a pound. (There are various ways to arrive at this solution, the most "primitive" of which is "counting".)

1. Arranging the slices so they can be divided into ***4*** equal groups of whole slices and a remainder group that can be divided into quarters of slices.

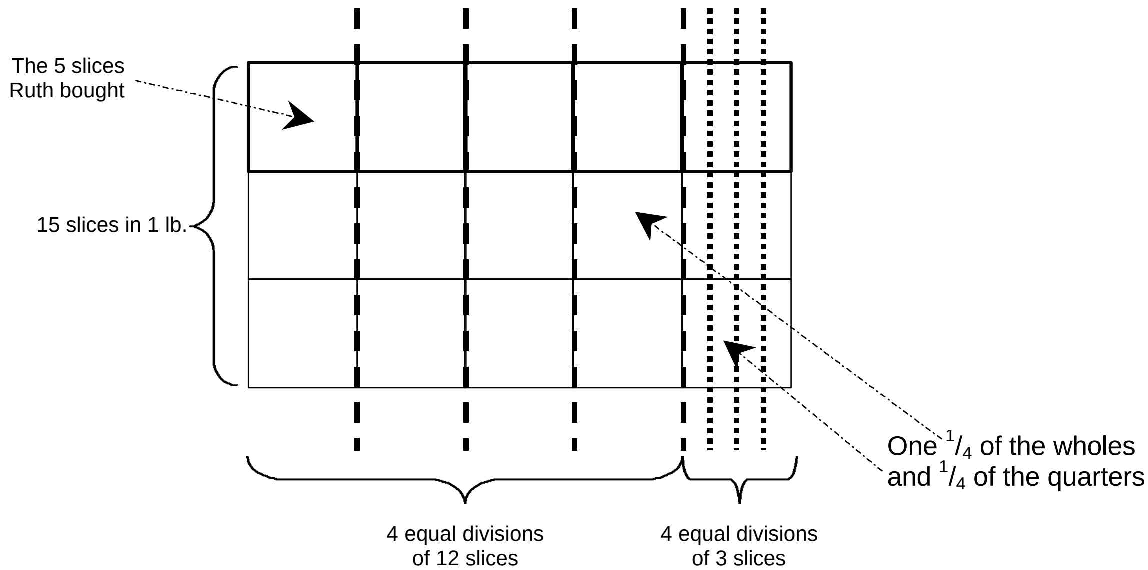
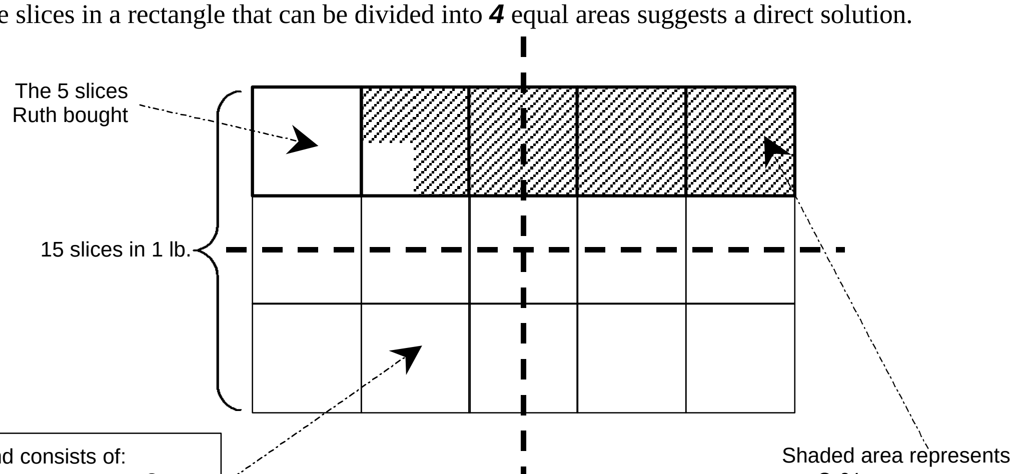


Figure 11. First Solution. Linear Slicing
Create ***4*** groups, each of which contains an equal number of slices. Then divide the remainder to ***4*** equal parts. Select one fourth of wholes and one fourth of the quarters.

2. Arranging the slices in a rectangle that can be divided into ***4*** equal areas suggests a direct solution.



Each ¼ of a pound consists of:

2	whole slices	=	2
3	½ slices	=	$^3/_2$
1	¼ slice	=	¼
Total		=	3 ¾

Figure 12. Second Solution. Two-dimensional Slicing

Arrange the 15 slices into a rectangle and divide it into 4 equal parts, each of which consists of $2 + 3 \times ½ + ¼ = 3¾$.

Indeed, both experiments confirm that the calculation is correct.

12. Decimal Fractions

There are two equivalent ways to define decimal fractions.

- The common definition:

  Decimal fractions *are fractions having only powers of ten as the denominators and expressed using the decimal place-value system.*

- The alternative definition:

  When integers are multiplied by powers of ten, the results are larger whole numbers. When integers are divided by powers of ten larger, the results are decimal fractions.

  - Advance definition. *When integers are multiplied by positive powers of ten, the results are larger whole numbers. When integers are multiplied by negative powers of ten, the results are decimal fractions.*

Note that the alternative definition does not address how decimals are expressed. When the definition of decimal fractions depends on ordinary fractions, which have a two-dimensional expression system, we must specifically note the fact that decimal fractions are different and are expressed differently.

The alternative definition is based on the fact that decimal fractions share the expression method that students are already familiar with — that which they use to express integers.

Decimal Fractions vs. Ordinary Fractions

Decimal fractions and ordinary fractions share an important property: They are numeric values that we use to describe quantities that are parts of the whole. But in many other and important respect they are quite different.

	Ordinary Fractions	Decimal Fractions
Similarities	They are numeric values that we use to describe quantities that are parts of the whole.	
Differences	Use a special two-dimensional notation system. Examples: $\frac{123}{1000}$, $45\frac{\;}{100}$, $\frac{123}{\;} + 45\frac{67}{\;}$ $45\frac{123\ +}{1000} = 45\frac{\;}{1000}$	we use for integers. ***0.123, 45.67*** ***0.123*** ***<u>45.67</u>*** ***45.793***
	The precise values of all numbers can be expressed with a finite number of digits.	It may be impossible to express the precise value of a number with a finite number of digits.
Advance Differences	Can express only rational numbers.	Can express both rational and irrational numbers.

Table 1. Decimal Fractions vs. Ordinary Fractions

Introduction to Decimals

Since we can define fractions from two different perspectives, we can introduce them at two different phases of the mathematical education.

- Introducing decimals after teaching ordinary fractions.
- Introducing decimals together with teaching the decimal place-value system.

Recommendation. Introduce decimals in the commonly practiced order but take advantage of their association with and similarities to integers and integer operations.

Introducing Decimals After Teaching Ordinary Fractions

Even when introducing decimal fractions after the students learned fractions we can take advantage of the fact that decimal numbers are a direct extension of the familiar way we write and express integers. To this end, we should use the same teaching material that is described for teaching decimals as part of the decimal place-value system.

Introducing decimals When Teaching Decimal Place-Value System

We base the introduction on associating decimal fractions with the *decimal place-value system* as oppose to ordinary fractions. In the context of applying powers of ten to integers, whether we increase the values by multiplication or decrease the value by division, the symmetry is highly intuitive. Student can grasp the concept of decimal fractions and how to use them with greater ease.

- Introduce decimals independently of fractions as an extension of the place-value system.
- Converge the two systems for fraction description only after the students are fluent in both.
  - Introduce percents as a different notation of the decimal, place value system.
    - The benefit — when the fraction is more important/significant than the whole, percents let us use integers to describe the fractions.
      — In a sense it introduces ***units*** to the number system. Because the notion of percents does to fractions what inches do to feet, feet to miles, ounces to pounds, centimeters to meters, etc.
  - Let student discover the relationships between the ordinary and decimal fractions on their own.

Introduction Percents

Introduce percents as soon as possible. In fact you can do that in the same lesson in which you introduce decimals.

We use percents when fraction based on one hundredth are more significant than the whole. So, as we have seen in lessons on measurements, measuring and units, it is convenient and make life easier (remember — mathematicians are lazy) to give these units a special name.

Percent is nothing but a "decimal unit" to describe the specific fraction of one hundredth.

A reminder. It is beneficial to invent special units when we often and regularly use small or large quantities. That is why we have cents (not only dollars), miles and inches (not only feet), light years, etc. Percent is the name of the special unit the was given to one hundredth. (In fact there exists an equivalent to the percent that is based on the one thousandth. It is called ***per mill*** or ***permil***, *meaning one part of a thousand*. It even has a special symbol: ‰. But ***per mill*** is rarely, if ever, used. (Note that the two words are still separated the way *percent* used to be written.)

"Numeric Fractals" — Visual Patterns in Numeric Fractions

- **Terminating fractions** — can be stopped any time with precise result.

$$0.5 =$$

$$\frac{1}{2} = \frac{1}{1+1} = \frac{1}{1+\frac{2}{2}} = \frac{1}{1+\frac{2}{1+1}} = \frac{1}{1+\frac{2}{1+\frac{2}{2}}} = \frac{1}{1+\frac{2}{1+\frac{2}{1+1}}} = \frac{1}{1+\frac{2}{1+\frac{2}{1+\frac{2}{2}}}} = \ldots = \frac{1}{1+\frac{2}{1+\frac{2}{1+\frac{2}{1+\frac{2}{1+1}}}}}$$

- **Non-terminating, repeating fractions** — can be stopped any time with precise result.

$$0.333\ldots =$$

$$\frac{1}{3} = \frac{1}{2+1} = \frac{1}{2+\frac{3}{3}} = \frac{1}{2+\frac{3}{2+1}} = \frac{1}{2+\frac{3}{2+\frac{2}{2}}} = \frac{1}{2+\frac{3}{2+\frac{3}{2+1}}} = \frac{1}{2+\frac{3}{2+\frac{3}{2+\frac{3}{3}}}} = \ldots = \frac{1}{2+\frac{3}{2+\frac{3}{2+\frac{3}{2+\frac{3}{2+1}}}}}$$

- **Non-terminating, non-repeating fractions** — can <u>**never**</u> be stopped with precise result.

$$\sqrt{2} =$$

$$= 1+\frac{1}{2+1} = 1+\frac{1}{2+\frac{1}{1}} = 1+\frac{1}{2+\frac{1}{2+1}} = 1+\frac{1}{2+\frac{1}{2+\frac{1}{1}}} = 1+\frac{1}{2+\frac{1}{2+\frac{1}{2+1}}} = \ldots = 1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\ldots}}}}}$$

13. Summary

Lesson Summary

- Fraction introduction
- Basic fraction operations — caparison, subtraction and addition.
- Fraction multiplication.

- Fraction Division.
- Decimal Fractions. (Optional. Another alternative is to associate decimal fractions with the decimal place value system.)
  - Percents

Summary of Math Teaching Methodology

- **Hands-on introduction.** Introduce fraction as a real-world, practical, hands-on things / properties of things.
- **Plunge right into the most general form of fraction operations.** Gradual introduction — first addition and then subtraction — and sequential dependency — first common denominators, then related denominators and only at the end unlike denominators — takes too much time. And, if student falls behind on one of these, he/she will stay behind as you move ahead or will slow-down your teaching. Experience showed me that the students are capable of understanding the general concepts if we present them properly. If you follow this scheme, like denominators (and other such concepts) are nothing more than special cases of the general theory and they fall into place. You will have more time to let your students explore fractions thereby discover on their own such things as the shortcuts that are the results of like denominators.
- Introduce decimals independently of fractions as an extension of the place-value system. Only later converge the two systems.
- Mathematics consists of many intertwined threads. Continuously exposing students to the interrelationships between the various topics improves their mathematical thinking. Also, those students who may not understand a topic from one angle, may gain a better understanding from another.
  - Fractions and fraction operations, including decimals and percents.
  - Integers.
  - Multiplication.
  - Geometry.
  - Mathematical reasoning and visual thinking.
- Benefit for teachers of all grades.

Teaching fraction this way you are more effective & efficient.

Your students learn deeper and sooner.

Your students exercise their mathematical reasoning more.