Greatest Common Factor, Finding

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Background

To find the ***greatest common factor, GCF,*** is useful for many mathematical operations especially when working with fractions.

The Common Method

When given two integers, m and n, the common method to find their GCF is to divide both numbers by the most obvious factors until all are exhausted. Then GCF is equal to the product of all of these factors. For example, if both m and n are even divide them by 2. If the results m_1 and n_1 are again even, divide again by 2. If other obvious factors are known, such as 3, 5, 9 or 11, then divide by these factors. The problem is that often no factor is easily apparent for both m and n.

Alternative Method

As far as I know this method, in its numerical-computational version, is a well known, at least among mathematicians, algorithm. I might even be not quite efficient one. However, I don't know of anyone using the graphic (geometric) version as a teaching tool, especially to visually-oriented students.

Concept

This alternative method does not rely on knowing any common factors. It employs simple division and each division step is progressively simpler. This method has graphical representation. When teaching visually-oriented students, who are experiencing difficulties understanding any computational method, employing drawing with this graphic method may be easier.

The Geometric (Graphic) Method

In general, you fit as many time as possible the small dimension into the larger one. You do it by constructing as many squares as possible starting on the smaller edge. Once the whole rectangle is filled, you are done. As long that there is a leftover rectangle, you repeat the process with respect to it.

Informally, this algorithm can be stated as follows.

> Given any two integers, m and n, If $m = n$, there is nothing to do, for the GCF = m. Otherwise, draw the m-by-n rectangle, which like any rectangle, it has a small edge and a large edge. Now, and as long as you have a rectangle with one edge longer than the other, draw the largest square that fits within the rectangle such that one edge of the square is the short edge of the rectangle. This process repeats itself until as a result of drawing such a square what is left over is also a square. The length of the edge of this last square is your GCF.

Formally, the procedure is stated as follows:

Given two numbers m and n, such that $m \geq n$. Find their GCF.

1. Draw an m by n rectangle.
2. Let $m_1 = m$ and let $n_1 = n$. From no on we will refer to m_1 instead of m and to n_1 instead of n. (The reason for this will become clear in Step 5.2 below.)
3. Measure n_1 along the long, the m_1 edge.
4. Create the $n_1 \times n_1$ square such that its first side is the n_1 edge and its perpendicular side is collinear with the m_1 edge.
5. Consider the area of the $m_1 \times n_1$ rectangle that is outside the $n_1 \times n_1$ square.

   5.1. If the $n_1 \times n_1$ square left no such area out of the $m_1 \times n_1$ rectangle, then $m_1 = n_1$ and therefore $m = n$. There is nothing more to do.

   5.2. If the $n_1 \times n_1$ square leaves a remaining rectangle out of the $m_1 \times n_1$ rectangle, then the sides of this remaining rectangle are n_1 and $m_1 – n_1$. From this point on the procedure becomes recursive. Therefore, we replace the indexes of m and n with **i**, (where **i** = 1, 2, 3,…) There are three possible alternatives:

   5.2.1. If $m_i – n_i > n_I$, then let $m_{i+1} = m_i – n_i >$ and let $n_{i+1} = n_I$. Repeat Steps 3 through 5.2 until the result of the test in Step 5.2 is either 5.2.2 or 5.2.3.

   5.2.2. If $m_i – n_i < n_i$, then l et $n_{i+1} = m_i – n_i >$ and let $m_{i+1} = n_I$. Repeat Steps 3 through 5.2 until the result of the test in Step 5.2 is 5.2.1 or 5.2.3.

5.2.3. If $m_i - n_i = n_i$, then go to Step 6.

6. GCF = n_i. In other words, the Greatest Common Factor is n_i. You are done.

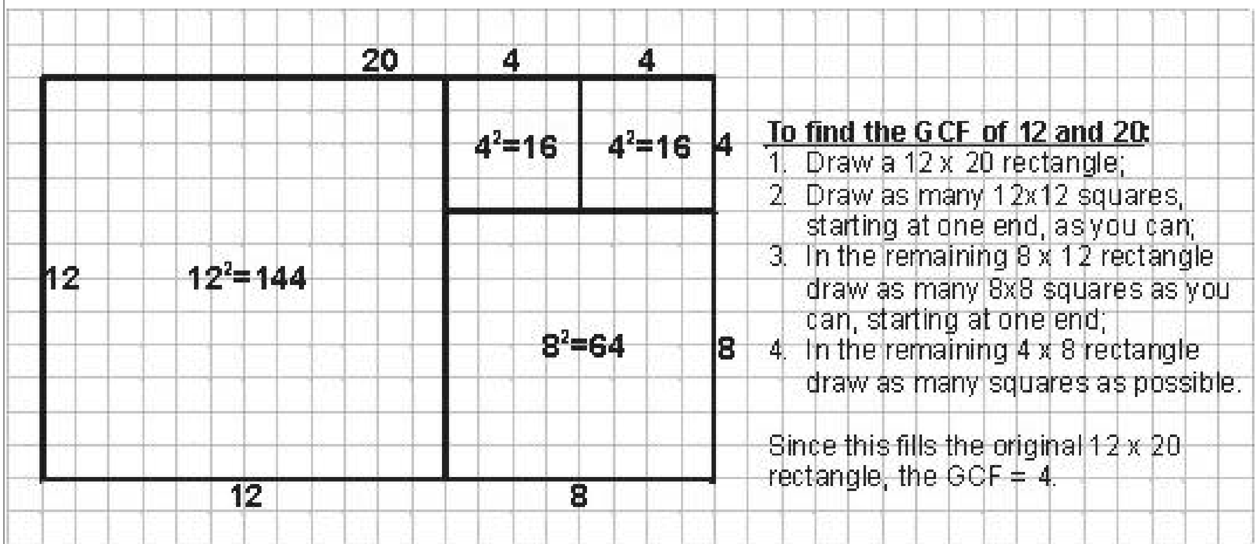

Figure 1. Finding the Greatest Common Factor of 12 and 20 Geometrically

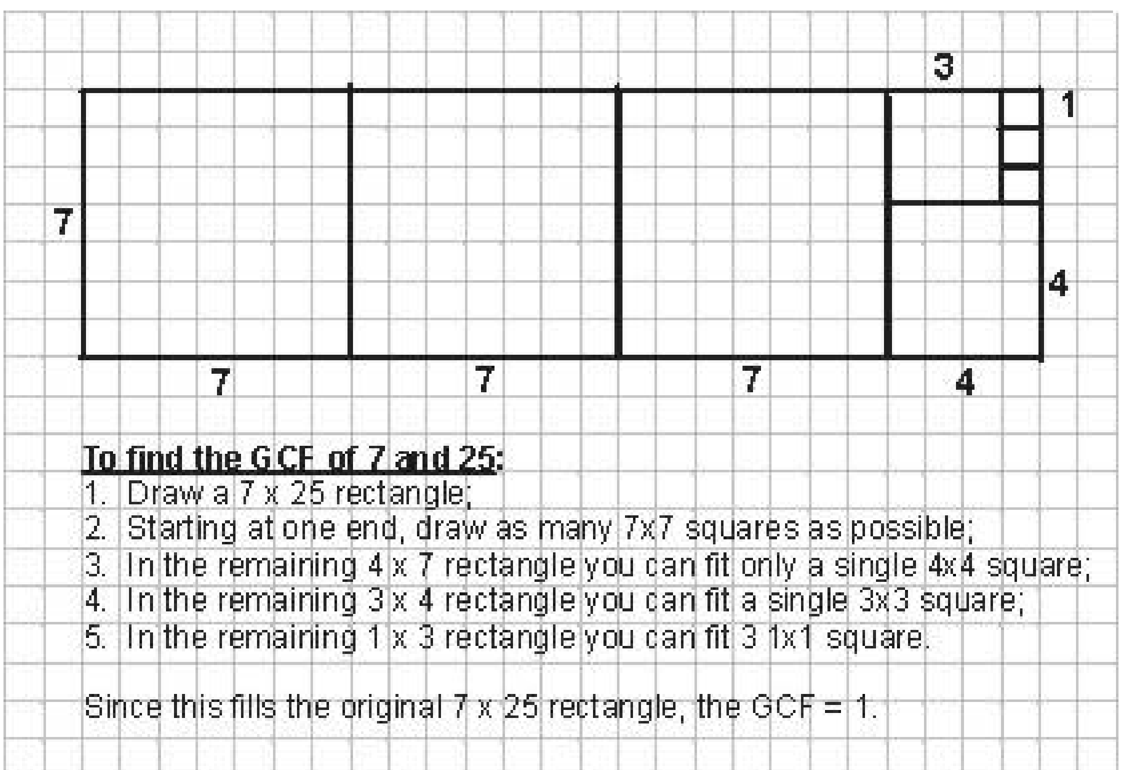

Figure 2. Finding the Greatest Common Factor of 7 and 25 Geometrically

The Computational (Numeric) Method

The computational method is the equivalent of the geometric one.

Given two integers, m and n, assume that $m < n$, to calculate their GCF, denoted by GCF_{mn}, then:

1. Let $m_1 = m$ and $n_1 = n$.
2. Let m_1' be largest multiple of m_1, such that $m_1' \leq n_1$.
3. If $m_1' = n_1$, then go to step 6.
4. If $m_i' < n_i$, then
   4.1. Let $m_{i+1} = n_i - m_i'$ (that is, $n_1 - m_1' = n_1$ modulo m_1) and
   4.2. $n_{i+1} = m_i$, where $i = 1, 2, 3, \ldots$
5. Go to step 2.
6. $GCF_{mn} = m_i$.

The procedure can be described using modulus arithmetic as follows:

1. Let $m_1 = m$ and $n_1 = n$.

2. Let $m_{i+1} = n_i$ modulo m_i, where $i = 1, 2, 3, \ldots$

If $m_{i+1} = 0$, then the process is complete. Go to Step *M5*.

If $m_{i+1} > 0$, then

2.1. $n_{i+1} = m_i$.

2.2. $i = i + 1$.

2.3. Go to step *M.2*.

$GCF_{mn} = m_i$.

Examples

In the computation sequence below r denotes the remainder.

- What is the GCF of *289* and *1,275*?

$$\frac{1275}{289} = 4,\ r = 119;\ \frac{289}{119} = 2,\ r = 51;\ \frac{119}{51} = 2,\ r = 17;\ \frac{51}{17} = 3,\ r = 0.$$

Answer: *17* (after 4 division steps)

- What is the GCF of *2,374,290* and *210,273*?

  Following the standard procedure, we don't get very far:

  $$\frac{2374290}{210273} \text{ (reduce by 3)} = \frac{7911430}{70091} \quad \text{Now what?}$$

  The problem is because the next and only other common factor is 31. The numbers 5 and 11 are excluded after following the available simple tests. The other candidate prime numbers before it are 7, 13, 17, 19, 23 and 29.

  Using alternative method, the process requires a sequence of long-division operations between numbers that grow progressively smaller.

  $$\frac{2374290}{210273} = 11,\ r = 61287;\ \frac{210273}{61287} = 3,\ r = 26412;\ \frac{61287}{26412} = 2,\ r = 8463;\ \frac{26412}{8463} = 3,\ r = 1023;$$

  $$\frac{8463}{1023} = 8,\quad r = 279;\quad \frac{1023}{279} = 3,\quad r = 186;\quad \frac{279}{186} = 1,\quad r = 93;\quad \frac{186}{93} = 2,\quad r = 0.$$

  Answer: *93* (after 8 division steps)